

# A symbolic approach to multiple zeta values at the negative integers

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## Abstract

Symbolic computation techniques are used to derive some closed form expressions for an analytic continuation of the Euler-Zagier zeta function evaluated at the negative integers as recently proposed in [1]. This approach allows to compute explicitly some contiguity identities, recurrences on the depth of the zeta values and generating functions.

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## 1 Introduction

The multiple zeta functions, first introduced by Euler and generalized by D. Zagier [2], appear in diverse areas such as quantum field theory [5] and knot theory [7]. These are defined by

$$\zeta_r(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}, \quad (1)$$

where  $\{n_i\}$  are complex values, and (1) converges when the constraints

$$\operatorname{Re}(n_r) \geq 1, \text{ and } \sum_{j=1}^k \operatorname{Re}(n_{r+1-j}) \geq k, \quad 2 \leq k \leq r, \quad (2)$$

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are satisfied (see [8]). Their values at integer points  $\mathbf{n} = (n_1, \dots, n_r)$  satisfying (2) are called *multiple zeta values*. An equivalent definition of these values is

$$\zeta_r(n_1, \dots, n_r) = \sum_{k_1 > 0, \dots, k_r > 0} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \dots (k_1 + \dots + k_r)^{n_r}}.$$

The sum of the exponents  $n_1 + \dots + n_r$  is called the *weight* of the zeta value, and the number  $r$  of these exponents is called its *depth*.

Following the result by Zhao [8] that the multiple zeta function has an analytic continuation to the whole space  $\mathbb{C}^r$ , several authors have recently proposed different analytic continuations based on a variety of approaches: Akiyama et al. [3] used the Euler-Maclaurin summation formula and Matsumoto [4] the Mellin-Barnes integral formula.

B. Sadaoui [1] provided recently such analytic continuation based on Raabe's identity, which links the multiple integral

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1, +\infty)^r} \frac{d\mathbf{x}}{(x_1 + a_1)^{n_1} (x_1 + a_1 + x_2 + a_2)^{n_2} \dots (x_1 + a_1 + \dots + x_r + a_r)^{n_r}}$$

to the multiple zeta function

$$Z(\mathbf{n}, \mathbf{z}) = \sum_{k_1 \geq 1, \dots, k_r \geq 1} \frac{1}{(k_1 + z_1)^{n_1} (k_1 + z_1 + k_2 + z_2)^{n_2} \dots (k_1 + z_1 + \dots + k_r + z_r)^{n_r}}$$

by

$$Y_0(\mathbf{n}) = \int_{[0, 1]^r} Z(\mathbf{n}, \mathbf{z}) d\mathbf{z}.$$

B. Sadaoui uses a classical inversion argument to obtain an analytic continuation of the multiple zeta function defined at negative integer arguments  $-\mathbf{n} = (-n_1, \dots, -n_r)$ . The argument uses the following three steps:

- the integral  $Y_{\mathbf{a}}(\mathbf{n})$  is computed for values of  $n_1, \dots, n_r$  that satisfy the convergence conditions (2),
- the values  $\mathbf{n}$  are replaced by  $-\mathbf{n}$  in this result: it is then shown that  $Y_{\mathbf{a}}(-\mathbf{n})$  is a polynomial in the variable  $\mathbf{a}$ ,
- the variables  $\mathbf{a} = (a_1, \dots, a_r)$  are replaced by  $(\mathcal{B}_1, \dots, \mathcal{B}_r)$ , and each Bernoulli symbol  $\mathcal{B}_k$  satisfies the two evaluation rules:

**evaluation rule 1:** each power  $\mathcal{B}_k^p$  of the Bernoulli symbol  $\mathcal{B}_k$  should be evaluated as

$$\mathcal{B}_k^p \rightarrow B_p, \tag{3}$$

the  $p$ -th Bernoulli number

**evaluation rule 2:** for two different symbols  $\mathcal{B}_k$  and  $\mathcal{B}_l$ ,  $k \neq l$ , the product  $\mathcal{B}_k^p \mathcal{B}_l^q$  is evaluated as

$$\mathcal{B}_k^p \mathcal{B}_l^q \rightarrow B_p B_q, \tag{4}$$

the product of the Bernoulli numbers  $B_p$  and  $B_q$ . If  $k = l$ , the first rule applies to give the evaluation

$$\mathcal{B}_k^p \mathcal{B}_k^q \rightarrow B_{p+q}.$$

**Example 1.** An example of depth 2, appearing in [1], is now computed using the rules above. The integral  $Y_{\mathbf{a}}(n_1, n_2)$  is explicitly computed and, replacing  $(n_1, n_2)$  by  $(-n_1, -n_2)$  gives

$$Y_{a_1, a_2}(-n_1, -n_2) = \frac{1}{n_2 + 1} \sum_{k_2=0}^{n_2+1} \sum_{l_1=0}^{n_1+n_2+2-k_2} \sum_{l_2=0}^{k_2} \frac{\binom{n_2+1}{k_2} \binom{n_1+n_2+2-k_2}{l_1} \binom{k_2}{l_2}}{n_1 + n_2 + 2 - k_2} a_1^{l_1} a_2^{l_2}.$$

Then substituting the variables  $a_1$  and  $a_2$  with the Bernoulli symbols  $\mathcal{B}_1$  and  $\mathcal{B}_2$  gives

$$\zeta_2(-n_1, -n_2) = \frac{1}{n_2 + 1} \sum_{k_2=0}^{n_2+1} \sum_{l_1=0}^{n_1+n_2+2-k_2} \sum_{l_2=0}^{k_2} \frac{\binom{n_2+1}{k_2} \binom{n_1+n_2+2-k_2}{l_1} \binom{k_2}{l_2}}{n_1 + n_2 + 2 - k_2} \mathcal{B}^{l_1} \mathcal{B}^{l_2}.$$

Using the evaluation rules (3) and (4) for the Bernoulli symbols, the multiple zeta value of depth 2 at  $(-n_1, -n_2)$  is

$$\zeta_2(-n_1, -n_2) = \frac{1}{n_2 + 1} \sum_{k_2=0}^{n_2+1} \sum_{l_1=0}^{n_1+n_2+2-k_2} \sum_{l_2=0}^{k_2} \frac{\binom{n_2+1}{k_2} \binom{n_1+n_2+2-k_2}{l_1} \binom{k_2}{l_2}}{n_1 + n_2 + 2 - k_2} B_{l_1} B_{l_2}.$$

The general case is given in [1, eq. (4.10)] as the  $(2r - 1)$ -fold sum<sup>1</sup>

$$\begin{aligned} \zeta_r(-n_1, \dots, -n_r) = & (-1)^r \sum_{k_2, \dots, k_r} \frac{1}{(\bar{n} + r - \bar{k})} \prod_{j=2}^r \frac{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}{k_i}} \quad (5) \\ & \times \sum_{l_1, \dots, l_r} \binom{\bar{n} + r - \bar{k}}{l_1} \binom{k_2}{l_2} \dots \binom{k_r}{l_r} B_{l_1} \dots B_{l_r} \end{aligned}$$

where  $k_2, \dots, k_r \geq 0$ ,  $l_j \leq k_j$  for  $2 \leq j \leq r$  and  $l_1 \leq \bar{n} + r + \bar{k}$  and

$$\bar{n} = \sum_{j=1}^r n_j, \quad \bar{k} = \sum_{j=2}^r k_j. \quad (6)$$

A symbolic expression for (5) is proposed here. This is used as a convenient tool to derive some specific zeta values at negative integers, contiguity identities for the multiple zeta functions, recursions on their depth and generating functions.

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1. This corrects a typo in [1, eq. (4.10)]

## 2 Main result

Introduce first the symbols  $\mathcal{C}_{1,2,\dots,k}$  defined recursively in terms of the Bernoulli symbols  $\mathcal{B}_1, \dots, \mathcal{B}_r$  as

$$\mathcal{C}_1^n = \frac{\mathcal{B}_1^n}{n}, \quad \mathcal{C}_{1,2}^n = \frac{(\mathcal{C}_1 + \mathcal{B}_2)^n}{n}, \dots \text{ and } \mathcal{C}_{1,2,\dots,k+1}^n = \frac{(\mathcal{C}_{1,2,\dots,k} + \mathcal{B}_{k+1})^n}{n}$$

with the symbolic computation rule:

**$\mathcal{C}$ -symbols rule:** All symbols  $\mathcal{C}_{1,2,\dots,k}$  are expanded using the above identities to express them only in terms of  $\mathcal{B}_k$ . The evaluation rules (3) and (4) for the Bernoulli symbols are then applied.

**Example 2.** For example,

$$\mathcal{C}_1^{n_1} \mathcal{C}_2^{n_2} = \mathcal{C}_1^{n_1} \frac{(\mathcal{C}_1 + \mathcal{B}_2)^{n_2}}{n_2} = \frac{1}{n_2} \sum_{k=0}^{n_2} \binom{n_2}{k} \mathcal{C}_1^{n_1+k} \mathcal{B}_2^{n_2-k} = \frac{1}{n_2} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{\mathcal{B}_1^{n_1+k}}{n_1+k} \mathcal{B}_2^{n_2-k}$$

is evaluated as

$$\frac{1}{n_2} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{B_{n_1+k}}{n_1+k} B_{n_2-k}.$$

The next result is given in terms of this notation.

**Theorem 2.1** *The multiple zeta values (5) at the negative integers  $(-n_1, \dots, -n_r)$  are given by*

$$\zeta_r(-n_1, \dots, -n_r) = \prod_{k=1}^r (-1)^{n_k} \mathcal{C}_{1,\dots,k}^{n_k+1}. \quad (7)$$

**Proof 2.2** *The inner sum in (5), in its Bernoulli symbols version,*

$$\sum_{l_1, \dots, l_r} \binom{\bar{n} + r - \bar{k}}{l_1} \binom{k_2}{l_2} \dots \binom{k_r}{l_r} \mathcal{B}^{l_1} \dots \mathcal{B}^{l_r},$$

can be summed to

$$(1 + \mathcal{B}_1)^{\bar{n}+r-\bar{k}} (1 + \mathcal{B}_2)^{k_2} \dots (1 + \mathcal{B}_r)^{k_r}.$$

The classical identity<sup>2</sup> for Bernoulli symbols  $\mathcal{B} + 1 = -\mathcal{B}$ , with  $\bar{n}$  defined in (6) reduces this to

$$(-1)^{\bar{n}+1} \mathcal{B}_1^{\bar{n}+r-\bar{k}} \mathcal{B}_2^{k_2} \dots \mathcal{B}_r^{k_r}. \quad (8)$$

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2. this identity can be deduced from the generating function

$$\exp(z\mathcal{B}) = \frac{z}{\exp(z) - 1}.$$

It follows that

$$\zeta_r(-\mathbf{n}) = \frac{(-1)^{\bar{n}}}{(n_r + 1)} \sum_{k_2, \dots, k_r} \mathcal{C}_1^{\bar{n}+r-\bar{k}} \mathcal{B}_2^{k_2} \dots \mathcal{B}_r^{k_r} \prod_{j=2}^r \frac{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}}.$$

Summing first over  $k_2$  gives

$$\zeta_r(-\mathbf{n}) = \frac{(-1)^{\bar{n}}}{(n_r + 1)} \sum_{k_3, \dots, k_r} \mathcal{C}_1^{n_1+1} \mathcal{C}_2^{n_2+\dots+n_r+r-1} \mathcal{B}_3^{k_3} \dots \mathcal{B}_r^{k_r} \prod_{j=3}^r \frac{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}}.$$

The result now follows by summing, in order, over the remaining indices.

Observe that the reduction (8) performed in the proof allows to restate a simpler version of Sadaoui's formula (5) as the more tractable  $(r-1)$ -fold sum

$$\zeta_r(-n_1, \dots, -n_r) = (-1)^{\bar{n}} \sum_{k_2, \dots, k_r} \frac{1}{(\bar{n} + r - \bar{k})} \prod_{j=2}^r \frac{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}{k_j} B_{l_1} \dots B_{l_r}}{\binom{\sum_{i=j}^r n_i + r - j + 1 - \sum_{i=j+1}^n k_i}} \quad (9)$$

Observe moreover that the derivation of (7) is unchanged if the symbols  $\mathcal{B}_1, \dots, \mathcal{B}_r$  are replaced by a generalization of the Bernoulli symbol  $\mathcal{B}$ , namely the polynomial Bernoulli symbol  $\mathcal{B} + z$  defined by

$$(\mathcal{B} + z)^n = B_n(z),$$

the Bernoulli polynomial of degree  $n$ . The same proof as above yields the next statement.

**Theorem 2.3** *The analytic continuation of the zeta function as given in [1] can be written as*

$$\zeta_r(-n_1, \dots, -n_r, z_1, \dots, z_r) = \prod_{i=1}^r \mathcal{C}_{1, \dots, i}^{n_i+1}(z_1, \dots, z_i) \quad (10)$$

with

$$\mathcal{C}_1^n(z_1) = \frac{(z_1 + \mathcal{B}_1)^n}{n} = \frac{B_n(z_1)}{n}, \quad \mathcal{C}_{1,2}^n(z_1, z_2) = \frac{(\mathcal{C}_1(z_1) + \mathcal{B}_2 + z_2)^n}{n}, \dots$$

and

$$\mathcal{C}_{1,2, \dots, k+1}^n(z_1, \dots, z_{k+1}) = \frac{(\mathcal{C}_{1,2, \dots, k}(z_1, \dots, z_k) + \mathcal{B}_{k+1} + z_{k+1})^n}{n}.$$

### 3 A general recursion formula on the depth

The methods above are now used to produce a general recursion formula on the depth of the zeta function.

**Theorem 3.1** *The multiple zeta functions satisfy the recursion rule*

$$\zeta_r(-\mathbf{n}; \mathbf{z}) = \frac{(-1)^{n_r}}{n_r + 1} \sum_{k=0}^{n_r+1} \binom{n_r+1}{k} (-1)^k \zeta_{r-1}(-n_1, \dots, -n_{r-1} - k; \mathbf{z}) B_{n_r+1-k}(z_r). \quad (11)$$

Introducing the new zeta symbol  $\mathcal{Z}_r$  with the evaluation rule <sup>3</sup>

$$\mathcal{Z}_r^k = \zeta_r(-n_1, \dots, -n_{r-1}, -n_r - k; \mathbf{z}),$$

this recursion rule can be written symbolically as

$$\zeta_r(-\mathbf{n}; \mathbf{z}) = (-1)^{n_r} \frac{(\mathcal{B} - \mathcal{Z}_{r-1})^{n_r+1}}{n_r + 1} = \zeta_1(-n_r; -\mathcal{Z}_{r-1}). \quad (12)$$

**Proof 3.2** *Start from (10) and expand the last term*

$$\mathcal{C}_{1,\dots,r}^{n_r+1}(z_1, \dots, z_r) = \frac{(\mathcal{C}_{1,\dots,r-1}^{n_r-1+1}(z_1, \dots, z_{r-1}) + \mathcal{B}_r(z_r))^{n_r+1}}{n_r + 1}$$

by using the binomial formula to produce

$$\begin{aligned} \zeta_r(-n_1, \dots, -n_r, z_1, \dots, z_r) &= \frac{(-1)^{n_r}}{n_r+1} \sum_{k=0}^{n_r+1} \binom{n_r+1}{k} \left( \prod_{i=1}^{r-2} \mathcal{C}_{1,\dots,i}^{n_i+1}(z_1, \dots, z_i) \right) \\ &\quad \times \mathcal{C}_{1,\dots,r-1}^{n_r+1+k}(z_1, \dots, z_{r-1}) \mathcal{B}_r^{n_r+1-k}(z_r). \end{aligned}$$

Then identify

$$\left( \prod_{i=1}^{r-2} \mathcal{C}_{1,\dots,i}^{n_i+1}(z_1, \dots, z_i) \right) \mathcal{C}_{1,\dots,r-1}^{n_r+1+k}(z_1, \dots, z_{r-1})$$

as

$$(-1)^{n_1+\dots+n_{r-2}+n_{r-1}+k} \zeta_{r-1}(-n_1, \dots, -n_{r-2}, -n_{r-1} - k; \mathbf{z})$$

to obtain the desired result.

Using the symbol  $\mathcal{Z}$ , this identity can be rewritten as

$$\zeta_r(-n_1, \dots, -n_r, z_1, \dots, z_r) = \frac{(-1)^{n_r}}{n_r+1} (\mathcal{B} - \mathcal{Z}_{r-1})^{n_r+1}$$

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3. note that  $\mathcal{Z}_r^0 \neq 1$

and the initial value

$$\zeta_1(-n; z) = (-1)^n \frac{(z + \mathcal{B})^{n+1}}{n+1}$$

provides the stated recursion.

#### 4 Contiguity identities

The multiple zeta function at negative integer values satisfies contiguity identities in the  $z$  variables. Two of them are presented here.

**Theorem 4.1** *The zeta function satisfies the contiguity identity*

$$\zeta_r(-n_1, \dots, -n_r; z_1, \dots, z_{r-1}, z_r + 1) = \zeta_r(-n_1, \dots, -n_r; z_1, \dots, z_{r-1}, z_r) + (-1)^{n_r} (z_r - \mathcal{Z}_{r-1})^{n_r}.$$

**Example 3.** In the case of the zeta function of depth 2,

$$\zeta_2(-n_1, -n_2, z_1, z_2 + 1) = \zeta_2(-n_1, -n_2, z_1, z_2) + (-1)^{n_1+1} (z_2 - \mathcal{Z}_1)^{n_2}$$

and the second term is expanded as

$$(-1)^{n_1+1} \sum_{k=0}^{n_2} \binom{n_2}{k} z_2^{n_2-k} (-1)^k \zeta_1(-n_1 - k; z_1).$$

**Proof 4.2** *Expand*

$$\begin{aligned} \zeta_r(-n_1, \dots, -n_r; z_1, \dots, z_{r-1}, z_r + 1) &= \frac{(-1)^{\bar{n}}}{n_r + 1} \mathcal{C}_1^{n_1+1}(z_1) \dots \mathcal{C}_{1, \dots, r-2}^{n_{r-2}+1}(z_1, \dots, z_{r-2}) \\ &\quad \times \sum_{k=0}^{n_r+1} \binom{n_r+1}{k} \mathcal{C}_{1, \dots, r-1}^{n_{r-1}+1+k}(z_1, \dots, z_{r-1}) B_{n_r+1-k}(z_r + 1). \end{aligned}$$

and use the identity on Bernoulli polynomials

$$B_{n_r+1-k}(z_r + 1) = B_{n_r+1-k}(z_r) + (n_r - k + 1) z_r^{n_r-k}$$

to produce the result.

The corresponding result for a shift in the first variable admits a similar proof.

**Theorem 4.3** *The depth-2 zeta function satisfies the contiguity identities*

$$\zeta_2(-n_1, -n_2, z_1 + 1, z_2) = \zeta_2(-n_1, -n_2, z_1, z_2) + \frac{(-1)^{n_1+n_2}}{n_2 + 1} z_1^{n_1} B_{n_2+1}(z_1 + z_2).$$

## 5 A Generating Function

The generating function of the zeta values at negative integers is defined by

$$F_r(w_1, \dots, w_r) = \sum_{n_1, \dots, n_r \geq 0} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} \zeta_r(-n_1, \dots, -n_r). \quad (13)$$

A recurrence for  $F_r$  is presented below. The initial condition is given in terms of the generating function for Bernoulli numbers

$$F_B(w) = \sum_{n=0}^{+\infty} \frac{B_n}{n!} z^n = \frac{w}{e^w - 1}.$$

**Theorem 5.1** *The generating function of the zeta values at negative integers satisfies the recurrence*

$$F_r(w_1, \dots, w_r) = \frac{1}{w_r} [F_{r-1}(w_1, \dots, w_{r-1}) - F_B(-w_r) F_{r-1}(w_1, \dots, w_{r-2}, w_{r-1} + w_r)]$$

with the initial value

$$F_1(w_1) = -\frac{1}{w_1} [e^{-w_1 B_1} - 1] = \frac{1 - F_B(-w_1)}{w_1}.$$

Moreover, the representation of the shift operator as  $\exp\left(a \frac{\partial}{\partial w}\right) \circ f(w) = f(w + a)$  and  $F_1(w, z) = -\frac{1}{w} [e^{-w(B+z)} - 1]$  give the recursion symbolically as

$$F_r(w_1, \dots, w_r) = F_1\left(w_r, -\frac{\partial}{\partial w_{r-1}}\right) \circ F_{r-1}(w_1, \dots, w_{r-1}),$$

so that

$$F_r(w_1, \dots, w_r) = F_1\left(w_r, -\frac{\partial}{\partial w_{r-1}}\right) \circ F_1\left(w_{r-1}, -\frac{\partial}{\partial w_{r-2}}\right) \circ \dots \circ F_1\left(w_2, -\frac{\partial}{\partial w_1}\right) \circ F_1(w_1).$$

**Proof 5.2** *Start from*

$$F_r(w_1, \dots, w_r) = \sum_{n_1, \dots, n_r} \frac{w_1^{n_1} \dots w_r^{n_r}}{n_1! \dots n_r!} (-1)^{n_1 + \dots + n_r} C_1^{n_1+1} \dots C_{1, \dots, r}^{n_r+1} = \prod_{j=1}^r C_{1, \dots, j} e^{-w_j C_{1, \dots, j}},$$



and expand

$$\mathcal{C}_{1,\dots,r} e^{-w_r \mathcal{C}_{1,\dots,r}} = \sum_{n=0}^{\infty} \frac{(-w_r)^n}{n!} \cdot \frac{(-1)^{n+1}}{n+1} (\mathcal{C}_{1,\dots,r-1} + \mathcal{B}_r)^{n+1} = -\frac{1}{w_r} \left( e^{-w_r(\mathcal{C}_{1,\dots,r-1} + \mathcal{B}_r)} - 1 \right),$$

to deduce that  $F_r(w_1, \dots, w_r)$  is

$$\begin{aligned} & \frac{1}{w_r} \left( \prod_{j=1}^{r-1} \mathcal{C}_{1,\dots,j} e^{-w_j \mathcal{C}_{1,\dots,j}} \right) - \frac{1}{w_r} \left( \prod_{j=1}^{r-2} \mathcal{C}_{1,\dots,j} e^{-w_j \mathcal{C}_{1,\dots,j}} \right) e^{-w_r \mathcal{B}_r} \mathcal{C}_{1,\dots,r-1} e^{-(w_{r-1} + w_r) \mathcal{C}_{1,\dots,r-1}} \\ &= \frac{1}{w_r} F_{r-1}(w_1, \dots, w_{r-1}) - \frac{1}{w_r} F_B(-w_r) F_{r-1}(w_1, \dots, w_{r-2}, w_{r-1} + w_r). \end{aligned}$$

## 6 Shuffle Identity

Multiple zeta values at positive integers satisfy *shuffle identities*, such as

$$\zeta_2(n_1, n_2) + \zeta_2(n_2, n_1) + \zeta_1(n_1 + n_2) = \zeta_1(n_1) \zeta_1(n_2).$$

The analytic continuation technique used in [1] does not preserve this identity at negative integers, while others do (for example, see [6]). The following theorem gives the correction terms.

**Theorem 6.1** *The zeta values at negative integers as defined in [1] satisfy the identity*

$$\zeta_2(-n_1, -n_2) + \zeta_2(-n_2, -n_1) + \zeta_1(-n_1 - n_2) - \zeta_1(-n_1) \zeta_1(-n_2) = \frac{(-1)^{n_1+1} n_1! n_2!}{(n_1 + n_2 + 2)!} B_{n_1+n_2+2}. \quad (14)$$

**Remark 1.** When  $n_1 + n_2$  is odd,  $B_{n_1+n_2+2} = 0$  so that the shuffle identity (14) holds for  $\zeta_2(-n_1, -n_2)$  as expected, since the depth-2 zeta function is holomorphic at these points.

**Proof 6.2** *Let  $\delta(w_1, w_2) = F_2(w_1, w_2) + F_2(w_2, w_1) + F_1(w_1 + w_2) - F_1(w_1) F_1(w_2)$ . An elementary calculation gives*

$$\delta(w_1, w_2) = \frac{\frac{1}{w_1} + \frac{1}{w_2} - \frac{1}{2} \coth\left(\frac{w_1}{2}\right) - \frac{1}{2} \coth\left(\frac{w_2}{2}\right)}{w_1 + w_2}.$$

*The expansions*

$$\frac{1}{w_1} - \frac{1}{2} \coth\left(\frac{w_1}{2}\right) = - \sum_{k=0}^{+\infty} \frac{w_1^{2k+1}}{(2k+2)!} B_{2k+2} \text{ and } \frac{1}{w_1 + w_2} = \frac{1}{w_2} \sum_{l \geq 0} \left(-\frac{w_1}{w_2}\right)^l$$

*now produce*

$$\delta(w_1, w_2) = - \sum_{k,l=0}^{+\infty} (-1)^l \frac{B_{2k+2}}{(2k+2)!} \left(w_1^{2k+l+1} w_2^{-l-1} + w_1^l w_2^{2k-l}\right).$$

*Identifying the coefficient of  $w_1^{n_1} w_2^{n_2}$  in this series expansion gives the result.*

## 7 Specific multiple zeta values

This final section gives some examples of the evaluation at negative integers of the zeta function, obtained from (5) and (12).

1: for depth  $r = 2$ ,

$$\zeta_2(-n, 0) = (-1)^n \left[ \frac{B_{n+2}}{n+2} - \frac{1}{2} \frac{B_{n+1}}{n+1} \right], \quad (15)$$

and

$$\zeta_2(0, -n) = \frac{(-1)^{n+1}}{n+1} [B_{n+1} + B_{n+2}]; \quad (16)$$

2: for depth  $r = 3$ ,

$$\zeta_3(-n, 0, 0) = \frac{(-1)^n}{2} \left[ \frac{B_{n+3}}{n+3} - 2 \frac{B_{n+2}}{n+2} + \frac{2}{3} \frac{B_{n+1}}{n+1} \right] \quad (17)$$

and

$$\zeta_3(0, -n, 0) = \frac{(-1)^{n+1}}{2} \left[ \frac{n}{(n+1)(n+2)} B_{n+2} - \frac{B_{n+1}}{n+1} + 2 \frac{B_{n+3}}{n+2} \right]. \quad (18)$$

3: as a final example, the recursion rule (11) is used to compute the value  $\zeta_3(0, 0, -2)$  as

$$\begin{aligned} \zeta_3(0, 0, -2) &= \frac{(\mathcal{B} - \mathcal{Z}_2)^3}{3} = \frac{1}{3} (\mathcal{B}^3 \mathcal{Z}_2^0 - 3\mathcal{B}^2 \mathcal{Z}_2^1 + 3\mathcal{B} \mathcal{Z}_2^2 - \mathcal{Z}_2^3) \\ &= \frac{1}{3} (B_3 \zeta_2(0, 0) - 3B_2 \zeta_2(0, -1) + 3B_1 \zeta_2(0, -2) - \zeta_2(0, -3)) = -\frac{1}{60}. \end{aligned}$$

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